ABCExceptions

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Chapter 1

Introduction

In this paper, we use tools from analytic number theory to estimate the number of triples of a given height satisfying the *abc* conjecture. Associated to any non-zero integer n is its radical

$$\mathrm{rad}(n) = \prod_{p|n} p.$$

We say that a triple $(a, b, c) \in \mathbb{N}^3$ with $\gcd(a, b, c) = 1$ is an *abc triple of exponent* λ if

$$a + b = c, \quad \mathrm{rad}(abc) < c^{\lambda}.$$

The well-known *abc* conjecture of Masser and Oesterlé asserts that, for any $\lambda < 1$, there are only finitely many *abc* triples of exponent λ. The best unconditional result is due to Stewart and Yu [12], who have shown that finitely many *abc* triples satisfy $\mathrm{rad}(abc) < (\log c)^{3-\varepsilon}$. Recently, Pasten [11] has proved a new subexponential bound, assuming that $a < c^{1-\varepsilon}$, via a connection to Shimura curves. In this paper we shall focus on counting the number $N_\lambda(X)$ of *abc* triples of exponent λ in a box $[1, X]^3$, as $X \to \infty$.

Definition 1.1. For $\lambda > 0$ define $N_\lambda(X)$ as the number of triples $(a, b, c) \in \mathbb{N}^3$ with $a + b = c$, $\gcd(a, b, c) = 1$ and $\mathrm{rad}(abc) < c^{\lambda}$.

Given $\lambda > 0$, an old result of de Bruijn [4] implies that

Lemma 1.2. *For any* $\varepsilon > 0$*, we have*

$$\#\left\{n \le x : \mathrm{rad}(n) \le x^{\lambda}\right\} \ll_{\varepsilon} x^{\lambda+\varepsilon}. \tag{1.0.1}$$

Proof.
It suffices to show that for any integer $k \ge 2$ we have

$$|\{n \le X : \ \mathrm{rad}(n) = k\}| \ll X^{O(1/\log\log X)}. \tag{1.0.2}$$

To prove (1.0.2), write $k = p_1 \cdots p_r$ as distinct primes $p_1 < \cdots < p_r$. Then $\mathrm{rad}(n) = k$ implies that $n = p_1^{m_1} \cdots p_r^{m_r}$ for some integers $m_1, \ldots, m_r \ge 1$. Therefore, the number of $n \le X$ in question is

$$\begin{aligned}
|\{n \le X : \ \mathrm{rad}(n) = k\}| &\le \sum_{m_1,\ldots,m_r \ge 1} \mathbf{1}_{\sum_{j\le r} m_j(\log p_j) \le \log X} \\
&\le \sum_{m_1,\ldots,m_r \ge 1} \int_{\mathbb{R}^r} \mathbf{1}_{\sum_{j\le r} t_j(\log p_j) \le \log X} \mathbf{1}_{t_j \in (m_j-1, m_j]\ \forall j \le r} \mathrm{d}t_1 \cdots \mathrm{d}t_r \\
&= \mathrm{vol}\Big(\{(t_1, \ldots, t_r) \in \mathbb{R}^r_{\ge 0} : \ t_1(\log p_1) + \cdots + t_r(\log p_r) \le \log X\}\Big) \\
&= (\log X)^r e^{O(r)} r^{-r} \prod_{j=1}^{r} \frac{1}{\log p_j}
\end{aligned}$$

by calculating the volume of a simplex. Moreover, we have

$$\prod_{j=1}^{r} \log p_j \ge \prod_{j=2}^{r+1} \log j = \exp\left(\sum_{j=2}^{r+1} \log\log j\right) \gg (\log r)^r e^{-O(r)}.$$

Hence for some $C_0 > 1$

$$|\{n \leq X : \ \mathrm{rad}(n) = k\}| \ll \left(\frac{C_0 \log X}{r \log r}\right)^r \ll \exp\left(O\Big(\frac{\log X}{\log\log X}\Big)\right),$$

since $r = \omega(k) \leq (1 + o(1))(\log X)/(\log\log X)$ by the prime number theorem.

□

Any triple (a, b, c) counted by $N_\lambda(X)$ must satisfy $\mathrm{rad}(abc) < X^\lambda$, and so we must have $\min\{\mathrm{rad}(a)\,\mathrm{rad}(b), \mathrm{rad}(b)\,\mathrm{rad}(c),$ $X^{2\lambda/3}$, since a, b, c are pairwise coprime. An application of (1.0.1) now leads to the following "trivial bound".

Proposition 1.3. *Let $\lambda > 0$. Then $N_\lambda(X) = O_\varepsilon(X^{2\lambda/3+\varepsilon})$, for any $\varepsilon > 0$.*

Proof. Any triple (a, b, c) counted by $N_\lambda(X)$ must satisfy $\mathrm{rad}(abc) < X^\lambda$, and so we must have $\min\{\mathrm{rad}(a)\,\mathrm{rad}(b), \mathrm{rad}(b)\,\mathrm{rad}$ $X^{2\lambda/3}$, since a, b, c are pairwise coprime. An application of 1.2 now leads to the following "trivial bound". □

The primary goal of this paper is to give the first power-saving improvement over this simple bound for values of λ close to 1.

Theorem 1.4. *Let $\lambda \in (0, 1.001)$ be fixed. Then $N_\lambda(X) = O(X^{33/50})$.*

Here we note that $33/50 = 0.66$. By comparison, the trivial bound in Proposition 1.3 would give $N_1(X) = O(X^{0.66\bar{6}+\varepsilon})$ and $N_{1.001}(X) = O(X^{0.6674})$. Moreover, we see that Theorem 1.4 gives a power-saving when $\lambda \in (0.99, 1.001)$. We emphasise that this power-saving represents a proof of concept of the methods; we expect that the exponent can be reduced with substantial computer assistance.

Theorem 1.4 also applies for λ slightly greater than 1, which places it in the realm of a question by Mazur [10]. Given a fixed $\lambda > 1$, he asked whether or not $N_\lambda(X)$ has exact order $X^{\lambda-1}$. In fact, Mazur studies the refined counting function

Definition 1.5. For $\alpha, \beta, \gamma > 0$, define $S_{\alpha,\beta,\gamma}(X)$ as the number of $(a, b, c) \in \mathbb{N}^3$ with $\gcd(a, b, c) = 1$ such that

$$a, b, c \in [1, X], \quad a + b = c, \quad \mathrm{rad}(a) \leq X^\alpha, \quad \mathrm{rad}(b) \leq X^\beta, \quad \mathrm{rad}(c) \leq X^\gamma.$$

The argument used to prove Proposition 1.3 readily yields

Lemma 1.6.

$$S_{\alpha,\beta,\gamma}(X) \ll_\varepsilon X^{\min\{\alpha+\beta, \alpha+\gamma, \beta+\gamma\}+\varepsilon}, \tag{1.0.3}$$

for any $\varepsilon > 0$.

Proof. □

Mazur then asks whether $S_{\alpha,\beta,\gamma}(X)$ has order $X^{\alpha+\beta+\gamma-1}$ if $\alpha + \beta + \gamma > 1$. Evidence towards this has been provided by Kane [9, Theorems 1 and 2], who proves that

$$X^{\alpha+\beta+\gamma-1-\varepsilon} \ll_\varepsilon S_{\alpha,\beta,\gamma}(X) \ll_\varepsilon X^{\alpha+\beta+\gamma-1+\varepsilon} + X^{1+\varepsilon},$$

for any $\varepsilon > 0$, provided that $\alpha, \beta, \gamma \in (0, 1]$ are fixed and satisfy $\alpha + \beta + \gamma > 1$. This result gives strong evidence towards Mazur's question when $\alpha + \beta + \gamma \geq 2$, but falls short of the trivial bound (1.0.3) when $\alpha + \beta + \gamma < 3/2$.

When considering *abc* triples of exponent $\lambda < 1$, we always have $\alpha+\beta+\gamma \leq \lambda < 1$, and the methods of Kane give no information in this regime. Indeed, we are not aware of any general estimates when $\lambda < 1$, beyond Proposition 1.3. Nonetheless, there do exist specific Diophantine equations which are covered by the *abc* conjecture and where bounds have been given for the number of solutions. For example, it follows from work of Darmon and Granville [5] that there are only finitely many coprime integer solutions to the Diophantine equation $x^p + y^q = z^r$, when $p, q, r \in \mathbb{N}$ are given and satisfy $1/p + 1/q + 1/r < 1$.

Proof outline

We now describe the main ideas behind the proof of Theorem 1.4. In terms of the counting function $S_{\alpha,\beta,\gamma}(X)$, our task is to show that whenever $\alpha, \beta, \gamma \in (0,1]$ satisfy $\alpha+\beta+\gamma \le \lambda$, we have $S_{\alpha,\beta,\gamma}(X) \ll X^{2\lambda/3-\eta}$, for some $\eta > 0$. A simple factorisation lemma (Proposition 2.6) will reduce the problem of bounding $S_{\alpha,\beta,\gamma}(X)$ to the problem of bounding the number of solutions to various Diophantine equations of the shape

$$\prod_{j\le d} x_j^j + \prod_{j\le d} y_j^j = \prod_{j\le d} z_j^j,$$

with specific constraints $x_i \sim X^{a_i}$, $y_i \sim X^{b_i}, z_i \sim X^{c_i}$ on the size of the variables, for admissible values of a_i, b_i, c_i (depending on α, β, γ). We then bound the number of solutions to these Diophantine equations using four different methods. The first of these (Proposition 3.1) uses Fourier analysis and Cauchy-Schwarz to estimate the number of solutions, leading to a bound that works well if two of the exponent vectors $(a_i)_i, (b_i)_i, (c_i)_i$ are somewhat "correlated". The second method (Proposition 3.2) uses the geometry of numbers and gives good bounds when one of a_1, b_1, c_1 is large. The remaining tools come from the determinant method of Heath-Brown (Proposition 3.14) and uniform upper bounds for the number of solutions to Thue equations (Proposition 3.15). For every choice of the exponents a_i, b_i, c_i we shall need to take the minimum of these bounds, which leads to a rather intricate combinatorial optimisation problem. This is solved by showing that at least one of the four methods always gives a power-saving over Proposition 1.3 when λ is close to 1.

Notation

We shall use $x \sim X$ to denote $x \in [X, 2X]$ and we put $[d] = \{1, \dots, d\}$. We denote by $\tau(n) = \sum_{d|n} 1$ the divisor function.

Chapter 2

Reduction to Diophantine equations

We will work with a variant of $S_{\alpha,\beta,\gamma}(X)$

Definition 2.1. Let $S^*_{\alpha,\beta,\gamma}(X)$ to be the number of $(a,b,c) \in \mathbb{N}^3$ with $\gcd(a,b,c)=1$ and

$$c \in [X/2, X], \quad a+b=c, \quad \mathrm{rad}(a) \sim X^{\alpha}, \quad \mathrm{rad}(b) \sim X^{\beta}, \quad \mathrm{rad}(c) \sim X^{\gamma}.$$

We begin by noting that by the pigeonhole principle,

Lemma 2.2. *We have*

$$N_{\lambda}(X) \ll (\log X)^4 \max_{\substack{\alpha,\beta,\gamma>0 \\ \alpha+\beta+\gamma\leq\lambda}} S^*_{\alpha,\beta,\gamma}(X). \tag{2.0.1}$$

Proof. □

Theorem 2.3. *There exists $\varepsilon > 0$ such that for all $\mathbf{c} \in \mathbb{Z}^3$ and $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{R}^d_{>0}$. we have*

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll X^{0.66-\varepsilon}. \tag{2.0.2}$$

Proof. □

Proof of Theorem 1.4. □

The following result allows us to bound $S^*_{\alpha,\beta,\gamma}(X)$ in terms of the number of solutions to certain monomial Diophantine equations. In order to state it, we need to introduce the quantity B_d

Definition 2.4. For $\mathbf{c} \in \mathbb{Z}^3$ and $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{R}^d_{>0}$. we have

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) := \# \left\{ (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbb{N}^{3d} \ : \ \begin{array}{l} x_i \sim X_i, \ y_i \sim Y_i, \ z_i \sim Z_i \\ c_1 \prod_{j\leq d} x_j^j + c_2 \prod_{j\leq d} y_j^j = c_3 \prod_{j\leq d} z_j^j \\ \gcd(c_1 \prod_{j\leq d} x_j, c_2 \prod_{j\leq d} y_j, c_3 \prod_{j\leq d} z_j) = 1 \end{array} \right\}. \tag{2.0.3}$$

Lemma 2.5. *Let $\varepsilon \in (0, 1/2)$, and let $2 \leq n \leq X$ be an integer. Then there exists a factorisation*

$$n = c \prod_{j \leq \frac{5}{2}\varepsilon^{-2}} x_j^j,$$

for positive integers x_j, c such that $c \leq X^{\varepsilon/2}$, the x_j are pairwise coprime, and

$$X^{-\varepsilon} \prod_{j \leq \frac{5}{2}\varepsilon^{-2}} x_j \leq \mathrm{rad}(n) \leq X^{\varepsilon} \prod_{j \leq \frac{5}{2}\varepsilon^{-2}} x_j.$$

Proof. Fix $2 \leqslant n \leqslant X$ and let $K = 2\lceil \varepsilon^{-1} \rceil$, $M = \lfloor \frac{5}{2}\varepsilon^{-2} \rfloor$. Define

$$y_j := \prod_{p^j \| n} p.$$

For $j \le M$, we set

$$x_j := \begin{cases} y_j & \text{for } j \neq K, \\ y_j \prod_{m>M} y_m^{\lfloor m/K \rfloor} & \text{for } j = K, \end{cases} \quad \text{and} \quad c := \prod_{m>M} y_m^{m-K\lfloor m/K \rfloor}.$$

All the x_j are pairwise coprime, since the y_j are pairwise coprime.

Note that by definition $c\prod_{j\le M} x_j^j = \prod_{m\ge 1} y_m^m = n \le X$. In particular,

$$\prod_{m\ge M} y_m \le \Big(\prod_{m\ge M} y_m^m\Big)^{1/M} \le X^{1/M}.$$

Then, since $m - K\lfloor m/K \rfloor \leqslant K$, it follows from the definition of c that

$$c \le \prod_{m\ge M} y_m^K \le X^{K/M} \le X^{\varepsilon/2}.$$

Thus

$$\operatorname{rad}(n) \le \operatorname{rad}(c) \prod_{j\le M} \operatorname{rad}(x_j) \le X^{\varepsilon/2} \prod_{j\le M} x_j.$$

On the other hand, we have

$$x_K = y_K \prod_{m>M} y_m^{\lfloor m/K \rfloor} \le \Big(y_K^K \cdot \prod_{m>M} y_m^m\Big)^{1/K} \le n^{1/K} \le X^{\varepsilon/2}.$$

Recalling that the y_j are squarefree and pairwise coprime for $j \neq K$, gives the lower bound

$$\operatorname{rad}(n) = \prod_{m\ge 1} y_m \ge \prod_{\substack{j\le M \\ j\neq K}} x_j \ge X^{-\varepsilon/2} \prod_{j\le M} x_j,$$

as claimed. □

Proposition 2.6. *Let $\alpha, \beta, \gamma \in (0, 1]$ be fixed and let $X \ge 2$. For any $\varepsilon > 0$ there exists an integer $d = d(\varepsilon) \ge 1$ such that the following holds. There exist $X_1, \dots, X_d, Y_1, \dots, Y_d, Z_1, \dots, Z_d \ge 1$ satisfying*

$$X^{\alpha-\varepsilon} \ll_\varepsilon \prod_{j=1}^{d} X_j \le 2X^{\alpha+\varepsilon}, \quad X^{\beta-\varepsilon} \ll_\varepsilon \prod_{j=1}^{d} Y_j \le 2X^{\beta+\varepsilon}, \quad X^{\gamma-\varepsilon} \ll_\varepsilon \prod_{j=1}^{d} Z_j \le 2X^{\gamma+\varepsilon} \tag{2.0.4}$$

and

$$\prod_{j=1}^{d} X_j^j \le X, \quad \prod_{j=1}^{d} Y_j^j \le X, \quad X^{1-\varepsilon^2} \ll_\varepsilon \prod_{j=1}^{d} Z_j^j \le X \tag{2.0.5}$$

and pairwise coprime integers $1 \le c_1, c_2, c_3 \le X^\varepsilon$, such that

$$S^*_{\alpha,\beta,\gamma}(X) \ll_\varepsilon X^\varepsilon B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}).$$

Proof of Proposition 2.6. We may assume that X is large enough in terms of ε, since otherwise the claim is trivial. Let (a, b, c) be a triple counted by $S^*_{\alpha,\beta,\gamma}(X)$. Apply Lemma 2.5 (with $\varepsilon^2/2$ in place of ε) to each of a, b, c to obtain factorisations of the form

$$a = c_1 \prod_{j\le d} x_j^j, \quad b = c_2 \prod_{j\le d} y_j^j, \quad c = c_3 \prod_{j\le d} z_j^j,$$

where $d = \lfloor 10\varepsilon^{-4} \rfloor$ and $1 \le c_1, c_2, c_3 \le X^{\varepsilon^2/4}$. Since (a, b, c) is counted by $S^*_{\alpha,\beta,\gamma}(X)$, we have $\gcd(a, b, c) = 1$ and $a+b = c$, so a, b, c are pairwise coprime. Hence, all the $3d+3$ numbers $x_i, y_i, z_i, c_1, c_2, c_3$ are pairwise coprime. Note also that by the properties of the factorisation given by Lemma 2.5, we have

$$X^{-\varepsilon/2} \prod_{j\le d} x_j \le \operatorname{rad}(a) \le X^{\varepsilon/2} \prod_{j\le d} x_j, \quad X^{-\varepsilon/2} \prod_{j\le d} y_j \le \operatorname{rad}(b) \le X^{\varepsilon/2} \prod_{j\le d} y_j,$$
$$X^{-\varepsilon/2} \prod_{j\le d} z_j \le \operatorname{rad}(c) \le X^{\varepsilon/2} \prod_{j\le d} z_j.$$

Since $\mathrm{rad}(a) \sim X^{\alpha}, \mathrm{rad}(b) \sim X^{\beta}, \mathrm{rad}(c) \sim X^{\gamma}$ for all triples under consideration, this implies

$$X^{\alpha-\varepsilon} \leq \prod_{j \leq d} x_j \leq X^{\alpha+\varepsilon}, \quad X^{\beta-\varepsilon} \leq \prod_{j \leq d} y_j \leq X^{\beta+\varepsilon}, \quad X^{\gamma-\varepsilon} \leq \prod_{j \leq d} z_j \leq X^{\gamma+\varepsilon}.$$

By dyadic decomposition, we can now find some X_i, Y_i, Z_i such that (2.0.4) and (2.0.5) hold, and such that

$$S^*_{\alpha,\beta,\gamma}(X) \ll_{\varepsilon} (\log X)^{3d} \sum_{\substack{\mathbf{c} \in \mathbb{N}^3 \\ c_1, c_2, c_3 \leq X^{\varepsilon/4}}} B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}).$$

Now the claim follows from the pigeonhole principle. □

Chapter 3

Upper bounds for integer points

3.1 Fourier analysis

The following result uses basic Fourier analysis to bound the quantity defined in (2.0.3).

Proposition 3.1 (Fourier analysis bound)**.** *Let $d \geq 1$, $\varepsilon > 0$ and $A \geq 1$ be fixed. Let*

$$X_1, \dots, X_d, Y_1, \dots, Y_d, Z_1, \dots, Z_d \geq 1$$

and put

$$\Delta = \max_{1 \leq i \leq d} (X_i Y_i Z_i). \tag{3.1.1}$$

Let $\mathbf{c} = (c_1, c_2, c_3) \in \mathbb{Z}^3$ satisfy $0 < |c_1|, |c_2|, |c_3| \leq \Delta^A$. Then

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^{\varepsilon} \frac{\prod_{j \leq d} \left(X_j Y_j Z_j (Y_j + Z_j)\right)^{\frac{1}{2}}}{\max_{i>1} \prod_{j \equiv 0 \bmod i} Z_j^{\frac{1}{2}}}.$$

Proof. By the orthogonality of characters, we have

$$\begin{aligned} B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) &\leq \int_0^1 \sum_{x_j \sim X_j} \sum_{y_j \sim Y_i} \sum_{z_j \sim Z_j} e\Big(\alpha\Big(c_1 \prod_{j \leq d} x_j^j + c_2 \prod_{j \leq d} y_j^j - c_3 \prod_{j \leq d} z_j^j\Big)\Big) \mathrm{d}\alpha \\ &= \int_0^1 S_1(\alpha) S_2(\alpha) S_3(-\alpha)\, \mathrm{d}\alpha, \end{aligned}$$

where

$$\begin{aligned} S_1(\alpha) &= \sum_{x_1 \sim X_1, \dots, x_d \sim X_d} e(\alpha c_1 x_1 x_2^2 \cdots x_d^d), \quad S_2(\alpha) = \sum_{y_1 \sim Y_1, \dots, y_d \sim Y_d} e(\alpha c_2 y_1 y_2^2 \cdots y_d^d), \\ S_3(\alpha) &= \sum_{z_1 \sim Z_1, \dots, z_d \sim Z_d} e(\alpha c_3 z_1 z_2^2 \cdots z_d^d). \end{aligned}$$

Then Cauchy-Schwarz gives

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \leq \left(\int_0^1 |S_1(\alpha)|^2\, \mathrm{d}\alpha\right)^{\frac{1}{2}} \left(\int_0^1 |S_2(\alpha)|^2 |S_3(\alpha)|^2\, \mathrm{d}\alpha\right)^{\frac{1}{2}} =: \sqrt{I_1 I_2}. \tag{3.1.2}$$

By Parseval's identity and the divisor bound, we have

$$\begin{aligned} I_1 = \int_0^1 |S_1(\alpha)|^2\, \mathrm{d}\alpha &= \sum_{x_j \sim X_j \forall j} \#\{(x_1', \dots, x_d') \colon x_j' \sim X_j\ \forall j,\ x_1 x_2^2 \cdots x_d^d = x_1' x_2'^2 \cdots x_d'^d\} \\ &\ll \prod_{j \leq d} X_j^{1+\varepsilon}, \end{aligned} \tag{3.1.3}$$

for any $\varepsilon > 0$. Using Cauchy-Schwarz again, for any $i \leqslant d$ we have

$$|S_3(\alpha)|^2 = \Bigg| \sum_{z_j \sim Z_j \, \forall j \le d} e(\alpha c_3 z_1 \cdots z_d^d) \Bigg|^2 \le T(\alpha) \prod_{j \not\equiv 0 \bmod i} Z_j,$$

where

$$T(\alpha) = \sum_{z_j \sim Z_j \, \forall j \not\equiv 0 \bmod i} \Bigg| \sum_{z_j \sim Z_j \, \forall j \equiv 0 \bmod i} e(\alpha c_3 z_1 \cdots z_d^d) \Bigg|^2.$$

Let ir be the largest multiple of i in $[1, d]$. Then

$$\begin{aligned} I_2 &= \int_0^1 |S_2(\alpha)|^2 |S_3(\alpha)|^2 \, \mathrm{d}\alpha \\ &\leqslant \prod_{j \not\equiv 0 \bmod i} Z_j \cdot \int_0^1 |S_2(\alpha)|^2 \, T(\alpha) \, \mathrm{d}\alpha \\ &= \prod_{j \not\equiv 0 \bmod i} Z_j \cdot \tilde{N}, \end{aligned} \tag{3.1.4}$$

where $\tilde{N}$ is the number of

$$(z_1, \dots, z_d, z_i', \dots, z_{ir}') \in \mathbb{N}^{d+r}, \quad (y_1, \dots, y_d) \in \mathbb{N}^d, \quad (y_1', \dots, y_d') \in \mathbb{N}^d$$

such that

$$y_j, y_j' \sim Y_j, \quad z_j \sim Z_j, \quad z_j' \sim Z_j$$

for all $j \le d$, and

$$c_3 \left(\prod_{j \equiv 0 \bmod i} z_j^j - \prod_{j \equiv 0 \bmod i} z_j'^j \right) \prod_{j \not\equiv 0 \bmod i} z_j^j + c_2 \prod_{j \le d} y_j^j - c_2 \prod_{j \le d} y_j'^j = 0.$$

Let us write

$$\tilde{N} = \tilde{N}_1 + \tilde{N}_2, \tag{3.1.5}$$

where $\tilde{N}_1$ is the contribution to $\tilde{N}$ from tuples with $\prod_{j \equiv 0 \bmod i} z_j^j = \prod_{j \equiv 0 \bmod i} z_j'^j$, and $\tilde{N}_2$ is the contribution of the complementary tuples.

Then by the divisor bound we have

$$\tilde{N}_1 = \#\Big\{ y_j, y_j' \sim Y_j, \, z_j \sim Z_j \, \forall j \le d \, : \, \prod_j y_j^j = \prod_j y_j'^j \Big\} \ll \prod_j Z_j Y_j^{1+\varepsilon}, \tag{3.1.6}$$

for any $\varepsilon > 0$. In order to bound $\tilde{N}_2$, we first note that $a - b \mid a^i - b^i$ for any integers $a \neq b$ and $i \geq 1$. Thus for any integers $n \neq 0$ and $i \geq 2$,

$$\#\{(a, b) \in \mathbb{Z}^2 : a^i - b^i = n\}| \le \tau(|n|) \max_{d \mid n} \#\{b \in \mathbb{Z} : (b + d)^i - b^i = n\} \ll_\varepsilon |n|^\varepsilon.$$

This follows from the divisor bound and the fact that $(x + d)^i - x^i - n$ is a polynomial of degree $i - 1$. (Importantly, this argument fails when $i = 1$, since then the polynomial $(x + n)^i - x^i - n$ is identically 0.) Hence, on appealing to the divisor bound, we obtain

$$\begin{aligned} \tilde{N}_2 &\ll \prod_{j \le d} Y_j^2 \cdot \max_{0 < |n| \le \Delta^{A+k^2}} \# \left\{ \begin{array}{l} (z_1, \dots, z_d, z_i', \dots, z_{ir}') \in \mathbb{N}^{d+r} : z_j \sim Z_j \, z_j' \sim Z_j' \, \forall j \\ c_3 (\prod_{j \equiv 0 \bmod i} z_j^j - \prod_{j \equiv 0 \bmod i} z_j'^j) \prod_{j \not\equiv 0 \bmod i} z_j^j = n \end{array} \right\} \\ &\ll \prod_{j \le d} Y_j^2 Z_j^{\varepsilon/2} \cdot \max_{0 < |n| \le \Delta^{A+k^2}} \#\Big\{ (a, b, c) \in \mathbb{N}^3 : \; c(a^i - b^i) = n \Big\} \\ &\ll \Delta^\varepsilon \prod_{j \le d} Y_j^2. \end{aligned} \tag{3.1.7}$$

Combining (3.1.4), (3.1.5), (3.1.6) and (3.1.7), we deduce that

$$\begin{aligned} I_2 &\ll \Delta^\varepsilon \prod_{j \not\equiv 0 \bmod i} Z_j \cdot \Big(\prod_j Y_j Z_j + \prod_j Y_j^2 \Big) \\ &\ll \Delta^\varepsilon \prod_{j \equiv 0 \bmod i} Z_j^{-1} \cdot \prod_j \Big(Y_j Z_j^2 + Y_j^2 Z_j \Big). \end{aligned} \tag{3.1.8}$$

Plugging (3.1.3) and (3.1.8) back into (3.1.2), we conclude that

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \le \sqrt{I_1 I_2} \ll \Delta^\varepsilon \prod_{j \equiv 0 \bmod i} Z_j^{-\frac{1}{2}} \cdot \prod_j (X_j Y_j Z_j (Y_j + Z_j))^{\frac{1}{2}},$$

which is the desired bound. □

3.2 Geometry of numbers

We can supplement Proposition 3.1 with the following bound, where $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is defined in (2.0.3).

Proposition 3.2 (Geometry of numbers bound). *Let $d \ge 1$ and $\varepsilon > 0$ be fixed, and let*

$$X_1, \dots, X_d, Y_1, \dots, Y_d, Z_1, \dots, Z_d \ge 1.$$

Let $\mathbf{c} \in (c_1, c_2, c_3) \in \mathbb{Z}^3$ have non-zero and pairwise coprime coordinates. Then for Δ as in (3.1.1),

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^\varepsilon \min_{I, I', I'' \subset [d]} \Big(\prod_{i \in I} X_i \prod_{i \in I'} Y_i \prod_{i \in I''} Z_i \Big) \Big(1 + \frac{\prod_{i \notin I} X_i^i \prod_{i \notin I'} Y_i^i \prod_{i \notin I''} Z_i^i}{\max\{|c_1| \prod_i X_i^i,\ |c_2| \prod_i Y_i^i,\ |c_3| \prod_i Z_i^i\}} \Big).$$

Proof. Take any sets $I, I', I'' \subset [d]$. Let $(x_1, \dots, x_d, y_1, \dots, y_d, z_1, \dots, z_d)$ be a tuple counted by $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$. We fix a choice of $x_i, y_i, z_i \in \mathbb{Z}$ for all indices i in I, I', I'', respectively, and define

$$\begin{aligned} a_1 &= c_1 \prod_{i \in I} x_i^i, & a_2 &= c_2 \prod_{i \in I'} y_i^i, & a_3 &= c_3 \prod_{i \in I''} z_i^i, \\ x &= \prod_{i \notin I} x_i^i, & y &= \prod_{i \notin I'} y_i^i, & z &= \prod_{i \notin I''} z_i^i, \\ X &= \prod_{i \notin I} X_i^i, & Y &= \prod_{i \notin I'} Y_i^i, & Z &= \prod_{i \notin I''} Z_i^i. \end{aligned}$$

Then $\gcd(a_1, a_2, a_3) = 1$ and $\gcd(x, y, z) = 1$. According to Heath-Brown [6, Lemma 3], the number of triples (x, y, z) that contribute to $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is

$$\ll\ 1 + \frac{XYZ}{\max\{|a_1|X,\ |a_2|Y,\ |a_3|Z\}}.$$

Moreover, by the divisor bound, any triple (x, y, z) corresponds to $O_\varepsilon(\Delta^\varepsilon)$ choices of x_i, y_j, z_r with $i \notin I, j \notin I', r \notin I''$. We arrive at the desired upper bound by summing over the choices $x_i, y_i, z_i \in \mathbb{Z}$, with i in I, I', I''. □

Theorem 3.3. *Let $\gcd(a_1, a_2, a_3) = 1$. Then for $X_1, X_2, X_3 > 1$, we have*

$$\#\left((x_1, x_2, x_3) \in \mathbb{Z}^3\ :\ \gcd(x_1, x_2, x_3) = 1,\ |x_i| \leqslant X_i,\ a_1 x_1 + a_2 x_2 + a_3 x_3 = 0\right) \ll\ 1 + \frac{X_1 X_2 X_3}{\max_i\{|a_i| X_i\}}.$$

Proof. □

3.3 Determinant method

In this section we will record a bound for $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ in (2.0.3) that proceeds via the determinant method of Bombieri-Pila [2] and Heath-Brown [8]. See [1] for a gentle introduction to the determinant method. We first record a basic fact about the irreducibility of certain polynomials.

Lemma 3.4. *Let $r \geq 1$ and let $g \in \mathbb{C}[x]$ be a polynomial which has at least one root of multiplicity 1. Then the polynomial $g(x) - y^r$ is absolutely irreducible.*

Proof. We may assume a factorisation $g(x) = l_1(x)^{e_1} \dots l_t(x)^{e_t}$, with pairwise non-proportional linear polynomials $l_1, \dots, l_t \in \mathbb{C}[x]$ and exponents $e_1, \dots, e_t \in \mathbb{N}$ such that $e_1 = 1$. But $\mathbb{C}[x]$ is a unique factorisation domain and so we can apply Eisenstein's criterion with the prime l_1 in order to deduce that $g(x) - y^r$ is irreducible over $\mathbb{C}[y]$. It then follows that $g(x) - y^r$ is irreducible over $\mathbb{C}$, as claimed in the lemma. □

Let p, q, r be positive integers and let $a_1, a_2, a_3 \in \mathbb{Z}_{\neq 0}$. We shall require a good upper bound for the counting function

$$N(X,Y,Z) = \#\left\{ (x,y,z) \in \mathbb{Z}^3_{\neq 0} : \begin{array}{l} |x| \leq X,\ |y| \leq Y,\ |z| \leq Z \\ \gcd(x,y) = \gcd(x,z) = \gcd(y,z) = 1 \\ a_1x^p + a_2y^q + a_3z^r = 0 \end{array} \right\},$$

for given $X, Y, Z \geq 1$. This is achieved in the following result.

Bombieri–Pila [2, Theorem 4]

Theorem 3.5. *Let $f(x)$ be a C^∞ function on a closed subinterval of $[0, N]$, and suppose that$F(x, f) = 0$, where $F(x, y) \in \mathbb{R}[x, y]$ is absolutely irreducible of degree $d \geqslant 2$. Suppose that $|f'(x)| \leqslant 1$. Then*

$$\#\{(x, f(x)) \in \{1, \dots, N\}^2\} \ll_d (\log N)^{O(N)} N^{1/d}$$

Proof. □

Heath-Brown Theorem 15

Theorem 3.6. *Let $F \in \mathbb{Z}[x_1, \dots, x_n]$ be an absolutely irreducible polynomial of degree d, and let $\varepsilon > 0$ and $B_1, \dots, B_n \geqslant 1$ be given. Define*

$$B = \max_{(e_1,\dots,e_n)} \Big(\prod_{i \leqslant n} B_i^{e_i} \Big)$$

where the maximum is taken over all integer n-tuples $(e_1, \dots, e_n)$ for which the corresponding monomial $x_1^{e_1} \cdots x_n^{e_n}$ occurs in F with non-zero coefficient.

Then there exists $D = D(n, d, \varepsilon)$ and an integer k with

$$k \ll_{n,d,\varepsilon} T^\varepsilon (\log \|F\|)^{2n-3} \exp\Big((n-1) \Big(\frac{\prod_{i \leqslant n} \log B_i}{\log B} \Big)^{1/(n-1)} \Big)$$

satisfying the following: There are k polynomials $F_1, \dots, F_k \in \mathbb{Z}[x_1, \dots, x_n]$ coprime to F, with $\deg F_i \leqslant D$, such that every root of $F(x_1, \dots, x_n) = 0$ with $x_i \leqslant B_i$ is also a root of $F_j(x_1, \dots, x_n) = 0$, for some $j \leqslant k$.

Proof. □

Theorem 3.7. *Given integers $n, a_1, a_2 \neq 0$, there are at most $O_{\varepsilon,D}(|na_1a_2X_1X_2|^\varepsilon)$ many solutions $(x_1, x_2) \in \mathbb{Z}^2$ such that $|x_i| \leqslant X_i$ and*

$$n = a_1x_1^2 + a_2x_2^2.$$

Proof. Suppose $n = a_1x_1^2 + a_2x_2^2$. Then $a_1n = a_1^2x_1^2 + a_1a_2x_2^2$. Let D be the squarefree part of a_1a_2. Then the number of solutions (x_1, x_2) with $|x_i| \leq X_i$ to this equations is at most the number of solutions (m_1, m_2) to $a_1n = m_1^2 + Dm_2^2$ with $|m_i| \leq X_ia_1a_2$. For any such solution, we have $m_1 + \sqrt{-D}m_2 \mid a_1n$ in $\mathbb{Q}(\sqrt{-D})$. The claim follows from the divisor bound in quadratic fields, in Lemma 3.8. □

Lemma 3.8. *Let $\varepsilon > 0$. Let $D \geqslant 1$ be a squarefree integer, and set $K = \mathbb{Q}(\sqrt{-D})$. Then for all $\alpha \in K$, the number of ideals dividing (α) in K is $O(N_\alpha^\varepsilon)$.*

Proof. Mimics the proof over the integers using the fundamental theorem of arithmetic, but with ideals [See link in comment] □

Theorem 3.9. *Given integers $n, a_1, a_2 \neq 0$ and $p \geqslant 3$, there are at most $O(p^{1+\omega(|n|)})$ many solutions $(x_1, x_2) \in \mathbb{Z}^2$ such that $|x_i| \leqslant X_i$ and*

$$n = a_1x_1^p + a_2x_2^p.$$

Proof. ☐

Definition 3.10. Let $\omega(n)$ denote the number of distinct prime factors of an integer n.

Lemma 3.11. *For any $n \geqslant 2$, we have $\omega(n) \ll \log(3n)/\log\log(3n)$.*

Proof. ☐

Lemma 3.12. *Let $\varepsilon > 0$ and $D \geq 1$ and assume that $p, q, r \in [1, D]$ are integers. Then*

$$N(X, Y, Z) \ll_{\varepsilon,D} Z \min\left(X^{\frac{1}{q}}, Y^{\frac{1}{p}}\right)(XY)^{\varepsilon},$$

where the implied constant only depends on ε and D. Furthermore, if $p = q \geq 2$, then we have

$$N(X, Y, Z) \ll_{\varepsilon,D} Z(|a_1a_2a_3|XYZ)^{\varepsilon}.$$

Proof. We fix a choice of non-zero integer $z \in [-Z, Z]$, of which there are $O(Z)$. When z is fixed, the resulting equation defines a curve in $\mathbb{A}^2$ and we can hope to apply work of Bombieri-Pila [2, Theorem 4], which would show that the equation has $O_{\varepsilon,D}(\max(X, Y)^{\frac{1}{\max(p,q)}+\varepsilon})$ integer solutions in the region $|x| \leq X$ and $|y| \leq Y$, where the implied constant only depends on ε and D. This is valid only when the curve is absolutely irreducible, which we claim is true when $z \neq 0$. But, for fixed $z \in \mathbb{Z}_{\neq 0}$ the polynomial $a_2y^q + a_3z^r$ has non-zero discriminant as a polynomial in y. Hence the claim follows from Lemma **??**. Rather than appealing to Bombieri-Pila, however, we can get a sharper bound by using work of Heath-Brown [8, Theorem 15]. For fixed $z \in \mathbb{Z}_{\neq 0}$ this gives the bound $O_{\varepsilon,D}(\min(X^{\frac{1}{q}}, Y^{\frac{1}{p}})(XY)^{\varepsilon})$ for the number of available x, y. ☐

Lemma 3.13. *Let $\varepsilon > 0$ and $D \geq 1$ and assume that $p = q, r \in [1, D]$ are integers. Then*

$$N(X, Y, Z) \ll_{\varepsilon,D} Z(|a_1a_2a_3|XYZ)^{\varepsilon}.$$

where the implied constant only depends on ε and D.

Proof. Suppose now that $p = q \geq 2$. Then, for given $z \in \mathbb{Z}_{\neq 0}$, we are left with counting the number of integer solutions to the equation $N = a_1x^p + a_2y^p$, with $|x|, |y| \leq \max(X, Y)$, and where $N = -a_3z^r$. For $p = 2$ this is a classical problem in quadratic forms. The bound $O_{\varepsilon,D}((|a_1a_2N|XY)^{\varepsilon})$ follows from Heath-Brown [7, Theorem 3], for example. For $p \geq 3$ we obtain a Thue equation. According to work of Bombieri and Schmidt [3], there are at most $O(p^{1+\omega(|N|)})$ solutions, for an absolute implied constant. Using the bound $\omega(|N|) \ll \log(3|N|)/(\log\log(3|N|))$, this is $O_{\varepsilon,D}((|a_3|Z)^{\varepsilon})$, which thereby completes the proof of the lemma. ☐

Using these lemmas we can now supplement Propositions 3.1 and 3.2 with further bounds for $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$, as defined in (2.0.3).

Proposition 3.14. *Let $d \geq 1$, and let*

$$X_1, \dots, X_d, Y_1, \dots, Y_d, Z_1, \dots, Z_d \geq 1.$$

Let $\mathbf{c} \in (c_1, c_2, c_3) \in \mathbb{Z}^3_{\neq 0}$. Then for Δ as in (3.1.1), we have

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^{\varepsilon} \prod_{j \leq d} X_jY_jZ_j \cdot \min_{p,q \geq 1}\left((X_pY_q)^{-1} \min\left(X_p^{\frac{1}{q}}, Y_q^{\frac{1}{p}}\right)\right).$$

Proof. Let $(x_1, \dots, x_d, y_1, \dots, y_d, z_1, \dots, z_d)$ be a tuple counted by $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$. For any integers $p, q \geq 1$, we fix all but x_p and y_q and apply the first part of Lemma 3.12. This gives

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^{\varepsilon} \prod_{\substack{j \leq d \\ j \neq p}} X_j \prod_{\substack{j \leq d \\ j \neq q}} Y_j \prod_{j \leq d} Z_j \cdot \min\left(X_p^{\frac{1}{q}}, Y_q^{\frac{1}{p}}\right),$$

from which the statement of the lemma easily follows. ☐

Proposition 3.15. *Let $d \geq 1$, and let*

$$X_1, \dots, X_d, Y_1, \dots, Y_d, Z_1, \dots, Z_d \geq 1.$$

Let $\mathbf{c} \in (c_1, c_2, c_3) \in \mathbb{Z}^3_{\neq 0}$. Then

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^\varepsilon \prod_{j \leq d} X_j Y_j Z_j \cdot \min_{p \geq 2} \prod_{\substack{j \leq d \\ p | j}} (X_j Y_j)^{-1},$$

where Δ is given by (3.1.1).

Proof. Let $(x_1, \dots, x_d, y_1, \dots, y_d, z_1, \dots, z_d)$ be a tuple counted by $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$. We collect together all indices which are multiples of p for any integer $p \geq 2$. Applying the second part of Lemma 3.13 now gives

$$B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ll \Delta^\varepsilon \prod_{\substack{j \leq d \\ j \not\equiv 0 \bmod p}} X_j Y_j Z_j \prod_{\substack{j \leq d \\ j \equiv 0 \bmod p}} Z_j,$$

from which the statement easily follows. □

Chapter 4

Combining the upper bounds

4.1 Preliminaries

Throughout this section, let $\varepsilon > 0$ be small but fixed. We now have everything in place to prove Theorem 1.4 for any

$$\lambda \in (0, 1 + \delta - \varepsilon).$$

and

$$\delta \leqslant 0.001 - \varepsilon.$$

We shall write δ as a symbol rather than its numerical value in order to clarify the argument. As will be clear from the proof, a somewhat larger value of δ would also work.

Our goal is to prove that the upper bound in (**??**) holds for $S^*_{\alpha,\beta,\gamma}(X)$, for any α, β, γ such that

$$\alpha + \beta + \gamma \leqslant \lambda < 1 + \delta - \varepsilon. \tag{4.1.1}$$

The following result allows us to limit the range of α, β, γ under consideration.

Proposition 4.1. *Let $\alpha, \beta, \gamma > 0$, and let $\varepsilon > 0$ be fixed. Then*

$$S^*_{\alpha,\beta,\gamma}(X) \ll X^{0.66-\varepsilon^2/2},$$

unless $\min\{\alpha + \beta, \beta + \gamma, \gamma + \alpha\} \geq 0.66 - \varepsilon^2$.

Proof. This is an immediate consequence of (1.0.3) (with $\varepsilon^2/2$ in place of ε). □

In the light of Proposition 2.6 (with ε^2 in place of ε), we wish to bound $B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})$ for any pairwise coprime integers $1 \leq |c_1|, |c_2|, |c_3| \leq X^{\varepsilon^2}$, any fixed $d \geq 1$, and any choice of $X_i, Y_i, Z_i \geq 1$, for $1 \leq i \leq d$ that satisfies (2.0.4) and (2.0.5). Moreover, α, β, γ satisfy (4.1.1).

It will be convenient to define

Definition 4.2. Define $a_i, b_i, c_i \in \mathbb{R}_{\geq 0}$ via

$$X_i = X^{a_i}, \quad Y_i = X^{b_i}, \quad Z_i = X^{c_i},$$

for $1 \leq i \leq d$, and $a_i = b_i = c_i = 0$ for $i > d$.

Lemma 4.3.

$$\sum_{i \leq d} i a_i \leq 1, \quad \sum_{i \leq d} i b_i \leq 1, \quad 1 - \varepsilon^2 \leq \sum_{i \leq d} i c_i \leq 1. \tag{4.1.2}$$

Proof. Follows from 2.6 □

In particular, in the light of (4.1.1) and Proposition 4.1, we may henceforth assume that

Lemma 4.4. *We have*

$$\sum_{i \leq d} (a_i + b_i) \geq 0.66 - \varepsilon^2, \quad \sum_{i \leq d} (a_i + c_i) \geq 0.66 - \varepsilon^2, \quad \sum_{i \leq d} (b_i + c_i) \geq 0.66 - \varepsilon^2 \tag{4.1.3}$$

and

$$\sum_{i \leq d} (a_i + b_i + c_i) \leq 1 + \delta - \varepsilon. \tag{4.1.4}$$

Proof. Follows from (4.1.1) and Proposition 4.1. □

Definition 4.5. It will be convenient to henceforth define

$$\nu = 2\varepsilon^2 + \frac{\log B_d(\mathbf{c}, \mathbf{X}, \mathbf{Y}, \mathbf{Z})}{\log X}.$$

Our goal is now to show that $\nu \leqslant 0.66$, since then (**??**) is a direct consequence of Proposition 2.6. This will then imply Theorem 1.4, via (2.0.1) and (**??**).

Before proceeding to the main tools that we shall use to estimate ν, we first show that we may assume that

$$0.32 - \delta \leq \sum_{i \leq d} a_i, \ \sum_{i \leq d} b_i, \ \sum_{i \leq d} c_i \leq 0.34 + \delta - \frac{1}{2}\varepsilon,$$

using the argument of Proposition 4.1.

Proposition 4.6.

$$0.32 - \delta \leq \sum_{i \leq d} a_i, \ \sum_{i \leq d} b_i, \ \sum_{i \leq d} c_i \leq 0.34 + \delta - \frac{1}{2}\varepsilon, \tag{4.1.5}$$

Proof. Indeed, suppose that $\sum_{i \leq d} c_i > 0.34 + \delta - \varepsilon/2$. Then (4.1.4) implies that

$$\sum_{i \leq d} (a_i + b_i) < 0.66 - \frac{1}{2}\varepsilon,$$

whence (1.0.3) yields $\nu < 0.66$. This shows that we may suppose that the upper bound in (4.1.5) holds. Suppose next that $\sum_{i \leq d} c_i < 0.32 - \delta$. Then, by the upper bound in (4.1.5), we have

$$\sum_{i \leq d} (b_i + c_i) < 0.66 - \frac{1}{2}\varepsilon,$$

which is again found to be satisfactory, via (1.0.3). □

Thus we may proceed under the assumption that the parameters a_i, b_i, c_i satisfy (4.1.4)-(4.1.5).

4.2 Summary of the main bounds

We now recast our bounds in Sections 3.1–3.3 in terms of an upper bound for ν, using the parameters a_i, b_i, c_i. In all of the following bounds, we may freely permute the exponent vectors $(a_i), (b_i), (c_i)$.

Proposition 4.7 (Fourier bound)**.**

$$\nu < \frac{1}{2}\Big(1 + \delta + \sum_{i \leq d} \max(a_i, b_i) - \max_{m > 1}(a_m, b_m)\Big).$$

Proof. It follows from Proposition 3.1 that

$$\nu \leq 3\varepsilon^2 + \frac{1}{2}\sum_{i \leq d}\Big(a_i + b_i + c_i + \max(a_i, b_i) - \max_{m>1} b_m\Big).$$

Permuting variables, the claim now follows from (4.1.4). □

Proposition 4.8 (Geometry bound)**.**

$$\nu < \delta + \min_{I, I', I'' \subset [d]} \left(\max\left(1\,,\, \sum_{i \in I} i a_i + \sum_{i \in I'} i b_i + \sum_{i \in I''} i c_i\right) - \sum_{i \in I} a_i - \sum_{i \in I'} b_i - \sum_{i \in I''} c_i\right).$$

Proof. Applying Proposition 3.2, we obtain

$$\nu \leq 3\varepsilon^2 + \min_{I, I', I''} \left(\sum_{i \notin I} a_i + \sum_{i \notin I'} b_i + \sum_{i \notin I''} c_i + \max\left(0\,,\, \sum_{i \in I} i a_i + \sum_{i \in I'} i b_i + \sum_{i \in I''} i c_i - \sum_{i \in [d]} i c_i\right)\right),$$

where the minimum runs over subsets $I, I', I'' \subset [d]$. Taking the lower bound $\sum_{i\in[d]} ic_i \geqslant 1-\varepsilon^2$, from (4.1.2), it follows that

$$\nu \le 4\varepsilon^2 + \min_{I,I',I''} \left(\sum_{i\notin I} a_i + \sum_{i\notin I'} b_i + \sum_{i\notin I''} c_i + \max\left(0, \sum_{i\in I} ia_i + \sum_{i\in I'} ib_i + \sum_{i\in I''} ic_i - 1\right)\right). \tag{4.2.1}$$

The proof now follows from (4.1.4). □

Proposition 4.9 (Determinant Bound)**.**

$$\nu < \min_{p,q\geqslant 1} \left(1+\delta - a_p - b_q + \min\left(\frac{a_p}{q}, \frac{b_q}{p}\right)\right).$$

Proof. Proposition 3.14 implies that

$$\nu \le 3\varepsilon^2 + \sum_{i\le d}(a_i+b_i+c_i) - \max_{p,q\ge 1}\left(\min\left(\frac{a_p}{q}, \frac{b_q}{p}\right) - a_p - b_q\right).$$

The claimed bound now follows from (4.1.4). □

Proposition 4.10 (Thue bound)**.**

$$\nu < 1+\delta - \max_{p\geqslant 2} \sum_{p|i} (a_i+b_i).$$

Proof. This easily follows from Proposition 3.15 and (4.1.4). □

4.3 Completion of the upper bound for ν

Assuming that $\delta \le 0.001$ and $\varepsilon > 0$ is sufficiently small, the remainder of this paper is devoted to a proof of the upper bound

$$\nu \leqslant 0.66,$$

for any choice of parameters a_i, b_i, c_i satisfying the properties recorded in (4.1.2)- (4.1.5). From this point onward, we will tacitly assume those properties hold.

Definition 4.11. It will be convenient to define constants $\delta_a, \delta_b, \delta_c$ via

$$\sum_{i\le d} a_i = \frac{1}{3} - \delta_a, \quad \sum_{i\le d} b_i = \frac{1}{3} - \delta_b, \quad \sum_{i\le d} c_i = \frac{1}{3} - \delta_c, \tag{4.3.1}$$

together with

$$\delta_{ab} := \delta_a + \delta_b, \quad \delta_{ac} := \delta_a + \delta_c, \quad \delta_{bc} := \delta_b + \delta_c,$$

and $\delta_s := \delta_a + \delta_b + \delta_c$.

Lemma 4.12. *We have*

$$\delta_{ab}, \delta_{ac}, \delta_{bc} \le 0.00\overline{6} + \varepsilon^2, \tag{4.3.2}$$

$$-0.00\overline{6} - \delta \le \delta_a, \delta_b, \delta_c \le 0.01\overline{3} + \delta + \varepsilon, \tag{4.3.3}$$

and

$$-\delta < \delta_s \le 0.01 + \varepsilon. \tag{4.3.4}$$

Proof. It follows from (4.1.3) that

$$\delta_{ab}, \delta_{ac}, \delta_{bc} \le 0.00\overline{6} + \varepsilon^2,$$

and from (4.1.5) that

$$-0.00\overline{6} - \delta \le \delta_a, \delta_b, \delta_c \le 0.01\overline{3} + \delta + \varepsilon,$$

and from (4.1.4) that $1 - \delta_s \le 1 + \delta$. Moreover, (4.3.2) implies $2\delta_s = \delta_{ab} + \delta_{ac} + \delta_{bc} \le 0.02 + 3\varepsilon^2$, so we must have

$$-\delta < \delta_s \le 0.01 + \varepsilon.$$

□

Definition 4.13. Define $s_i := a_i + b_i + c_i$.

Referring to (4.1.2), it will be convenient to record the inequalities

$$\sum_{i\geqslant 2}(i-1)a_i \leqslant \frac{2}{3} + \delta_a, \quad \sum_{i\geqslant 3}(i-2)a_i \leqslant \frac{1}{3} + a_1 + 2\delta_a, \quad \sum_{i\geqslant 4}(i-3)a_i \leqslant 2a_1 + a_2 + 3\delta_a, \tag{4.3.5}$$

that follow by subtracting. Similar relations hold for b_i and c_i, and thus for s_i.

Proposition 4.14. *To show $\nu \leq 0.66$, it suffices to assume*

$$a_j + b_j,\, a_j + c_j,\, b_j + c_j < 0.34 + \delta, \tag{4.3.6}$$

for each $j \geq 2$, and moreover,

$$a_2 + a_4 + b_2 + b_4,\;\; a_2 + a_4 + c_2 + c_4,\;\; b_2 + b_4 + c_2 + c_4 < 0.34 + \delta. \tag{4.3.7}$$

Hence

$$s_5, s_3, s_2 + s_4 < 0.51 + \frac{3}{2}\delta. \tag{4.3.8}$$

Proof. By the Thue bound, we have

$$\nu < 1 + \delta - \max_{p\geqslant 2}\sum_{p|i}(a_i + b_i),$$

and similarly for $a_i + c_i$ and $b_i + c_i$. Now (4.3.6) follows by taking $p = j$ and restricting the sum to $i = j$, while (4.3.7) follows by taking $p = 2$ and restricting $i \leq 4$.

Finally, (4.3.6) and (4.3.7) imply

$$s_5, s_3, s_2 + s_4 < \frac{3}{2}(0.34 + \delta) \leqslant 0.51 + \frac{3}{2}\delta. \tag{4.3.9}$$

□

Lemma 4.15. *We may assume*

$$s_1 + s_2 \leq 0.34 + \delta. \tag{4.3.10}$$

Proof. If $s_1 + s_2 > 0.34 + \delta$ then the Geometry bound and (4.3.8) imply that

$$\begin{aligned}\nu &\leqslant \max\left(1,\, s_1 + 2s_2\right) - s_1 - s_2 + \delta \\ &= \max(1 - s_1 - s_2, s_2) + \delta \\ &< \max\left(0.66, 0.51 + 3\delta\right) = 0.66.\end{aligned}$$

Thus we may proceed under the premise that

$$s_1 + s_2 \leq 0.34 + \delta.$$

□

Lemma 4.16. *For any $j \geqslant 3$, allow τ_j to be an element*

$$\tau_j \in \{a_j, b_j, c_j, s_j, a_j + b_j, a_j + c_j, b_j + c_j\}. \tag{4.3.11}$$

Then

$$\tau_j \in \left(0.34 - s_1 - s_2 + \delta, \frac{0.66 - s_2 - \delta}{j-1}\right) \implies \nu < 0.66 \tag{4.3.12}$$

and

$$\tau_3 \in \left(0.34 - s_1 + \delta, 0.33 - \frac{1}{2}\delta\right) \implies \nu < 0.66. \tag{4.3.13}$$

Proof. The Geometry bound gives

$$\begin{aligned}\nu &\leqslant \max\left(1, s_1 + 2s_2 + j\tau_j\right) - s_1 - s_2 - \tau_j + \delta \\ &= \max\left(1 - s_1 - s_2 - \tau_j, s_2 + (j-1)\tau_j\right) + \delta.\end{aligned}$$

Thus we have $\nu < 0.66$ if $\tau_j \in (0.34 - s_1 - s_2 + \delta, \frac{0.66 - s_2 - \delta}{j-1})$. In particular when $j = 3$, we have $\nu < 0.66$ if $\tau_3 \in (0.34 - s_1 - s_2 + \delta, 0.33 - \frac{1}{2}s_2 - \frac{\delta}{2})$. Similarly, by the Geometry bound, we have

$$\begin{aligned}\nu &\leqslant \max\left(1, s_1 + 3\tau_3\right) - s_1 - \tau_3 + \delta \\ &= \max\left(1 - s_1 - \tau_3, 2\tau_3\right) + \delta.\end{aligned}$$

Thus $\nu < 0.66$ if $\tau_3 \in (0.34 - s_1 + \delta, 0.33 - \frac{\delta}{2})$. ◻

Lemma 4.17. *We have*

$$a_3 \geqslant \frac{1}{3} - 4\delta_a - 3a_1 - 2a_2, \tag{4.3.14}$$

$$a_3 \geqslant \frac{1}{3} - \frac{5}{2}\delta_a - 2a_1 - \frac{3}{2}a_2 - \frac{1}{2}a_4.$$

Therefore

$$s_3 \geqslant 1 - 4\delta_s - 3s_1 - 2s_2, \tag{4.3.15}$$

and

$$s_3 \geqslant 1 - \frac{5}{2}\delta_s - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4. \tag{4.3.16}$$

Proof. Note that (4.3.5) gives $\sum_{i\geqslant 4} a_i \leqslant \sum_{i\geqslant 4}(i-3)a_i \leqslant 2a_1 + a_2 + 3\delta_a$. Similarly, we have $\sum_{i\geqslant 5} a_i \leqslant \frac{1}{2}\sum_{i\geqslant 5}(i-3)a_i \leqslant \frac{1}{2}(2a_1 + a_2 - a_4 + 3\delta_a)$. These imply that

$$a_3 = \frac{1}{3} - \delta_a - a_1 - a_2 - \sum_{i\geqslant 4} a_i \geqslant \frac{1}{3} - 4\delta_a - 3a_1 - 2a_2,$$

and

$$a_3 = \frac{1}{3} - \delta_a - a_1 - a_2 - a_4 - \sum_{i\geqslant 5} a_i \geqslant \frac{1}{3} - \frac{5}{2}\delta_a - 2a_1 - \frac{3}{2}a_2 - \frac{1}{2}a_4.$$

Analogous bounds hold for b_3 and c_3, and so we obtain

$$\begin{aligned}s_3 &\geqslant 1 - 4\delta_s - 3s_1 - 2s_2, \\ s_3 &\geqslant 1 - \frac{5}{2}\delta_s - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4.\end{aligned}$$

◻

We shall need to split the argument according to whether $s_2 < 0.3$ or $s_2 \geq 0.3$. Without loss of generality, we shall assume that $a_3 \geqslant b_3 \geqslant c_3$ in all that follows.

Case 1: Assume $s_2 \geqslant 0.3$.

Lemma 4.18. *If $s_2 \geq 0.3$,*

$$s_1 \leq 0.04 + \delta \tag{4.3.17}$$

and $s_4 < 0.21 + \frac{3}{2}\delta$.

Proof. Note that (4.3.10) gives

$$s_1 \leq 0.34 - s_2 + \delta \leqslant 0.04 + \delta, \tag{4.3.18}$$

and (4.3.8) gives

$$s_4 < 0.51 - s_2 + \frac{3}{2}\delta \leqslant 0.21 + \frac{3}{2}\delta.$$

◻

We further split into subcases.

Subcase 1.1: Assume $b_3 \leqslant 0.34 - s_1 - s_2 + \delta$.

Lemma 4.19. *$\nu \leq 0.66$ in the case $s_2 \geq 0.3$ and $b_3 \leqslant 0.34 - s_1 - s_2 + \delta$.*

Proof. Then

$$\begin{aligned} b_3 + c_3 \leqslant 2b_3 &\leqslant 2(0.34 - s_1 - s_2 + \delta) \\ &\leqslant 0.68 - 2s_2 + 2\delta \\ &\leq 0.33 - \frac{1}{2}s_2 - \frac{1}{2}\delta, \end{aligned}$$

for $s_2 \geqslant 0.3$ and $\delta \leq 0.001$. Hence, in view of (4.3.13), we may assume $b_3 + c_3 \leqslant 0.34 - s_1 - s_2 + \delta$. But then it follows from (4.3.16), (4.3.8) and (4.3.17) that

$$\begin{aligned} a_3 = s_3 - (b_3 + c_3) &\geqslant 1 - \frac{5}{2}\delta_a - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4 - (0.34 - s_1 - s_2 + \delta) \\ &= 0.66 - \frac{5}{2}\delta_a - s_1 - \frac{1}{2}(s_2 + s_4) - \delta \\ &\geq 0.66 - \frac{5}{2}\delta_a - (0.04 + \delta) - \frac{1}{2}(0.51 + \frac{3}{2}\delta) - \delta \\ &\geqslant 0.365 - \frac{5}{2}\delta_a - 3\delta. \end{aligned}$$

But (4.3.1) implies that $\frac{1}{3} - \delta_a \geqslant a_3 \geq 0.365 - \frac{5}{2}\delta_a - 3\delta$. Thus (4.3.3) implies that

$$0.031\bar{6} < \frac{3}{2}\delta_a + 3\delta \leq \frac{3}{2}(0.01\bar{3} + \delta + \varepsilon) + 3\delta \leqslant 0.02 + 5\delta.$$

This contradicts our assumption $\delta \leq 0.001$. □

Subcase 1.2: Assume $b_3 > 0.34 - s_1 - s_2 + \delta$.

Lemma 4.20. *$\nu \leq 0.66$ in the case $s_2 \geq 0.3$ and $b_3 > 0.34 - s_1 - s_2 + \delta$.*

Proof. By (4.3.13) we may assume

$$b_3 \geq 0.33 - \frac{1}{2}s_2 - \frac{1}{2}\delta. \tag{4.3.19}$$

By permuting the variables in (4.3.5), we have

$$\sum_{i \geqslant 4}(i-2)b_i \leqslant \frac{1}{3} - b_3 + b_1 + 2\delta_b.$$

We also have $b_1 \leq s_1 \leq 0.34 - s_2 + \delta$ by (4.3.10). Thus

$$\begin{aligned} \sum_{i \geq 4} b_i \leqslant \frac{1}{2}\left(\frac{1}{3} + b_1 - b_3 + 2\delta_b\right) &\leqslant \frac{1}{2}\Big(\frac{1}{3} + 0.34 - s_2 + \delta - (0.33 - \frac{s_2}{2} - \frac{\delta}{2}) + 2\delta_b\Big) \\ &< 0.33 - \frac{1}{2}s_2 - \frac{\delta}{2}, \end{aligned}$$

since (4.3.3) ensures that $\delta_b \leq 0.01\bar{3} + \delta + \varepsilon$ (and we have $\delta \leq 0.001$). A fortiori the same bound holds for $\sum_{i \geq 4} a_i$. Thus, in the light of (4.3.13), taking $\varepsilon > 0$ small we may assume that

$$a_4, b_4, a_5, b_5, a_6, b_6 \leq 0.34 - s_1 - s_2 + \delta.$$

Now write $M_i = \max(a_i, b_i)$ and $m_i = \min(a_i, b_i)$, so that $m_i + M_i = a_i + b_i$. By the Fourier bound we have

$$\nu < \frac{1}{2}\Big(1 + \delta + \sum_{i \leq d} \max(a_i, b_i) - \max(a_2, b_2)\Big) = \frac{1}{2}\Big(1 + \delta + \sum_{i \neq 2} M_i\Big).$$

On using (4.3.1), this implies that

$$\begin{aligned}2\nu - 1 - \delta < \sum_{i\neq 2} M_i &\leqslant \sum_{2\neq i\leqslant 6} M_i + \sum_{i\geqslant 7}(a_i+b_i)\\ &= \sum_{2\neq i\leqslant 6} M_i + \frac{2}{3} - \delta_{ab} - \sum_{i\leqslant 6}(a_i+b_i)\\ &= \frac{2}{3} - \delta_{ab} - \sum_{2\neq i\leqslant 6} m_i - (a_2+b_2).\end{aligned}$$

Next we lower bound a_2+b_2. To do this, we observe that by (4.3.5) we have

$$4\sum_{i\geqslant 7} a_i \leqslant \sum_{i\geqslant 7}(i-3)a_i = 2a_1 + a_2 + 3\delta_a - a_4 - 2a_5 - 3a_6,$$

whence

$$\frac{1}{3} - \delta_a = \sum_i a_i \leqslant \sum_{i\leqslant 6} a_i + \frac{1}{4}(2a_1 + a_2 + 3\delta_a - a_4 - 2a_5 - 3a_6) = \frac{1}{4}\sum_{i\leqslant 6}(7-i)a_i + \frac{3}{4}\delta_a.$$

Thus $a_2 \geqslant \frac{4}{15} - \frac{1}{5}\sum_{2\neq i\leqslant 6}(7-i)a_i - \frac{7}{5}\delta_a$, and similarly $b_2 \geqslant \frac{4}{15} - \frac{1}{5}\sum_{2\neq i\leqslant 6}(7-i)b_i - \frac{7}{5}\delta_b$. Since $m_3 = b_3$, it now follows that

$$\begin{aligned}2\nu - 1 - \delta &< \frac{2}{3} - \delta_{ab} - \sum_{2\neq i\leqslant 6} m_i - \Big(\frac{8}{15} - \frac{1}{5}\sum_{2\neq i\leqslant 6}(7-i)(a_i+b_i) - \frac{7}{5}\delta_{ab}\Big)\\ &\leqslant \frac{2}{15} + \frac{2}{5}\delta_{ab} + \frac{1}{5}\Big(6M_1 + m_1 + 4a_3 - b_3 + 3M_4 + 2M_5 + M_6\Big).\end{aligned} \tag{4.3.20}$$

Thus, using (4.3.19) and the bound $a_3 + b_3 \leq 0.34 + \delta$ coming from (4.3.6), we have

$$\begin{aligned}4a_3 - b_3 &\leqslant 4(0.34 - b_3 + \delta) - b_3\\ &< 4(0.01 + \frac{s_2}{2} + \frac{3\delta}{2}) - (0.33 - \frac{s_2}{2} - \frac{\delta}{2})\\ &= \frac{5}{2}s_2 - 0.29 + \frac{13\delta}{2}.\end{aligned}$$

Also $6M_1 + m_1 \leqslant 6s_1$ and recall $M_4, M_5, M_6 \leqslant 0.34 - s_1 - s_2 + \delta$. Hence plugging back into (4.3.20), we conclude

$$\begin{aligned}2\nu - 1 - \delta &< \frac{2}{15} + \frac{2}{5}\delta_{ab} + \frac{1}{5}\Big(6s_1 + (\frac{5}{2}s_2 - 0.29 + \frac{13\delta}{2}) + 6(0.34 - s_1 - s_2 + \delta)\Big)\\ &< \frac{2}{15} + \frac{2}{5}\delta_{ab} + \frac{1}{5}\Big(1.75 - \frac{7}{2}s_2 + 13\delta\Big)\\ &\leqslant 0.48\overline{3} - \frac{7}{10}s_2 + \frac{2}{5}\delta_{ab} + \frac{13}{5}\delta\\ &\leqslant 0.48\overline{3} - \frac{7}{10}(0.3) + \frac{2}{5}(0.00\overline{6} + \varepsilon^2) + \frac{13}{5}\delta < 0.279,\end{aligned}$$

since $s_2 \geq 0.3$, $\delta \leqslant 0.001$, and (4.3.2) implies that $\delta_{ab} \leq 0.00\overline{6} + \varepsilon^2$. Hence $\nu \leqslant \frac{1.3}{2} = 0.65$, which is more than satisfactory. □

Lemma 4.21. *$\nu \leq 0.66$ in the case $s_2 \geq 0.3$.*

Proof. Immediate from Lemmas 4.19 and 4.20. □

Case 2: Assume $s_2 < 0.3$.

Lemma 4.22.

$$b_3 < 0.17 + \frac{\delta}{2}. \tag{4.3.21}$$

Proof. It follows from (4.3.6) that

$$2b_3 \leqslant a_3 + b_3 < 0.34 + \delta, \tag{4.3.22}$$

whence $b_3 < 0.17 + \frac{\delta}{2}$. □

Lemma 4.23.

$$a_3 \geq 0.32 - 4\delta_s - s_1 - 2\delta. \tag{4.3.23}$$

Proof. We have $0.17 + \frac{\delta}{2} \leq 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$ since $s_2 < 0.3$ and $\delta \leq 0.001$. Thus, in view of (4.3.13), we may assume that $b_3, c_3 \leq 0.34 - s_1 - s_2 + \delta$. Then (4.3.15) gives

$$\begin{aligned} a_3 = s_3 - (b_3 + c_3) &\geqslant 1 - 4\delta_s - 3s_1 - 2s_2 - 2(0.34 - s_1 - s_2 + \delta) \\ &= 0.32 - 4\delta_s - s_1 - 2\delta. \end{aligned}$$

□

We shall proceed by separately handing the subcases

$$\mathbf{2.1}: \ a_3 \geq 0.32 \qquad\qquad \mathbf{2.2}: \ b_3 + c_3 < 0.33 - \frac{s_2}{2} - \frac{\delta}{2}.$$

These will be instrumental to proving the following subcases

$$\begin{aligned} &\mathbf{2.3}: \ 4s_1 + 3s_2 > 0.71, &\qquad &\mathbf{2.5}: \ 0.066 \leq s_2 \leq 0.204, \\ &\mathbf{2.4}: \ 4s_1 + \ s_2 < 0.4, &\qquad &\mathbf{2.6}: \ 2s_1 - s_2 > 0.025. \end{aligned}$$

Handling these subcases will complete the proof.

Lemma 4.24. *Assuming 4.27, 4.28, 4.29, 4.30, $\nu \leq 0.66$ in the case $s_2 < 0.3$.*

Proof. Indeed **2.3, 2.4, 2.6** each define half-planes that cover $[0,1]^2 \setminus T$, for the closed triangle T with vertices

$$(s_1, s_2) \in \{(0.06125, 0.155), (0.0785, 0.132), (0.0708\bar{3}, 0.11\bar{6})\}.$$

But then **2.5** covers T. Hence subcases **2.3–2.6** will complete the proof of Case 2. □

Subcase 2.1: Assume $a_3 \geq 0.32$

Lemma 4.25. *$\nu \leq 0.66$ in the case $s_2 < 0.3$ and $a_3 \geq 0.32$.*

Proof.

By (4.3.6) we have $b_3, c_3 \leqslant 0.34 + \delta - a_3 \leqslant 0.02 + \delta$. Let $m_i = \min(b_i, c_i)$, $M_i = \max(b_i, c_i)$, and $t_i = b_i + c_i = m_i + M_i$. If $M := \max_{i \geqslant 4} M_i > \frac{3}{4}(0.09)$, then using $\delta \leq 0.001$ and the Determinant bound (with variables permuted) yields

$$\nu \leqslant 1 + \delta - a_3 - M + \min\left(\frac{M}{3}, \frac{a_3}{4}\right) \leqslant 1 + \delta - a_3 - \frac{2}{3}M \leqslant 1 + \delta - 0.32 - \frac{1}{2}(0.09) \leqslant 0.636.$$

This is satisfactory. We may therefore assume that $M_i \leqslant \frac{3}{4}(0.09)$ for $i \geqslant 4$. Then $t_i \leqslant 2M_i \leqslant 0.135$ for $i \geqslant 4$. Moreover, $\sum_i (i-1)t_i \leqslant \frac{4}{3} + \delta_{bc}$, by (4.1.2) and (4.3.1). Appealing to the Geometry bound in the form (4.2.1), we deduce that

$$\nu \leqslant \varepsilon + a_3 + b_3 + m_4 + \sum_{i \geqslant 5} t_i + \max\Big(0, \sum_i is_i - 3(a_3 + b_3) - 4m_4 - \sum_{i \geqslant 5} it_i - 1\Big).$$

Thus we have $\nu \leqslant \max(\nu_1, \nu_2) + \varepsilon$, where

$$\nu_1 := a_3 + b_3 + m_4 + \sum_{i \geqslant 5} t_i \qquad \text{and} \qquad \nu_2 := \sum_i is_i - 2(a_3 + b_3) - 3m_4 - \sum_{i \geqslant 5} (i-1)t_i - 1.$$

Using (4.3.5), we see that

$$\begin{aligned} \nu_1 &\leqslant a_3 + b_3 + m_4 + t_5 + \frac{1}{5}\sum_{i \geqslant 6} (i-1)t_i \\ &\leqslant a_3 + b_3 + \frac{1}{2}t_4 + t_5 + \frac{1}{5}\Big(\frac{4}{3} + \delta_{bc} - t_2 - 2t_3 - 3t_4 - 4t_5\Big). \end{aligned}$$

Using $a_3 + b_3 \leqslant 0.34 + \delta$ (which follows from (4.3.21)), $\delta_{bc} < 0.00\bar{6} + \varepsilon^2$, and $t_5 \leqslant 0.135$, we conclude that

$$\nu_1 \leqslant a_3 + b_3 + \frac{1}{5}\Big(\frac{4}{3} + \delta_{bc} + t_5\Big) \leqslant 0.34 + \delta + \frac{1}{5}\Big(\frac{4}{3} + 0.00\bar{6} + \varepsilon^2 + 0.135\Big) < 0.637.$$

Similarly, on recalling $\sum_i ia_i \leqslant 1$, we have $\sum_i is_i - 1 \leqslant \sum_i it_i$, whence

$$\begin{aligned}\nu_2 &\leqslant \sum_i it_i - \sum_{i\geqslant 5}(i-1)t_i - 2(a_3+b_3) - 3m_4 \\ &= \sum_i t_i + \sum_{i\leqslant 4}(i-1)t_i - 2(a_3+b_3) - 3m_4 \\ &= \frac{2}{3} - \delta_{bc} + t_2 - 2(a_3 - c_3) + 3M_4,\end{aligned}$$

by (4.3.1). Using $t_2 \leqslant s_2 < 0.3$, $c_3 \leqslant 0.02 + \delta$, and $a_3 \geqslant 0.32$ by assumption, we conclude that

$$\nu_2 < \frac{2}{3} - \delta_{bc} + 0.3 - 2(0.3-\delta) + 3\cdot\frac{3}{4}(0.09) < 0.57 - \delta_{bc} + 2\delta.$$

Thus $\nu_2 < 0.6$, since (4.3.3) implies that $\delta_{bc} \geq -0.01\bar{3} - 2\delta$, and $\delta \leq 0.001$. Combining the bounds for ν_1 and ν_2, we conclude that $\nu \leqslant \max(\nu_1, \nu_2) + \varepsilon < 0.64$, which suffices. □

Subcase 2.2: Assume $b_3 + c_3 < 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$.

Lemma 4.26. *$\nu \leq 0.66$ in the case $s.2 < 0.3$ and $b_3 + c_3 < 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$.*

Proof. Then by (4.3.13) we may assume $\tau_3 = b_3 + c_3 < 0.34 - s_1 - s_2 + \delta$. By (4.3.16) we have

$$\begin{aligned}a_3 = s_3 - (b_3 + c_3) &> 1 - \frac{5}{2}\delta_s - 2s_1 - \frac{3}{2}s_2 - \frac{1}{2}s_4 - (0.34 - s_1 - s_2 + \delta) \\ &= 0.66 - \frac{5}{2}\delta_s - s_1 - \frac{1}{2}(s_2 + s_4) - \delta.\end{aligned}$$

It follows from (4.3.4) that $\delta_s \leq 0.01 + \varepsilon$ and from (4.3.8) that $s_2 + s_4 < 0.51 + 3\delta/2$. Hence

$$a_3 > 0.66 - \frac{5}{2}(0.01+\varepsilon) - s_1 - \frac{1}{2}(0.51 + \frac{3\delta}{2}) - \delta \geqslant 0.38 - s_1 - 3\delta.$$

Since $\delta \leqslant 0.001$, we see that $a_3 > 0.34 - s_1 + \delta$. Thus it follows from (4.3.13) that we may assume $\tau_3 = a_3 > 0.33 - \frac{\delta}{2} \geq 0.32$. Hence Subcase **2.1** completes the proof. □

Subcase 2.3: Assume $4s_1 + 3s_2 > 0.71$.

Lemma 4.27. *$\nu \leq 0.66$ in the case $s.2 < 0.3$ and $4s_1 + 3s_2 > 0.71$.*

Proof. Then the inequalities $b_3, c_3 \leqslant 0.34 - s_1 - s_2 + \delta$ give

$$b_3 + c_3 < 0.68 - 2(s_1 + s_2) + 2\delta < 0.325 - \frac{s_2}{2} + 2\delta.$$

Since $\delta \leq 0.001$, we see that $b_3 + c_3 < 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$. Hence Subcase **2.2** completes the proof. □

Subcase 2.4: Assume $4s_1 + s_2 < 0.4$.

Lemma 4.28. *$\nu \leq 0.66$ in the case $s.2 < 0.3$ and $4s_1 + s_2 < 0.4$.*

Proof. In this case, (4.3.6) and (4.3.23) give

$$b_3, c_3 \leqslant 0.34 - a_3 + \delta \leqslant 0.34 - (0.32 - 4\delta_s - s_1 - 2\delta) + \delta = 0.02 + 4\delta_s + s_1 + 3\delta.$$

In view of (4.3.4) and our assumption $4s_1 + s_2 < 0.4$, we deduce that

$$b_3 + c_3 \leqslant 0.12 + 8\varepsilon + 2s_1 + 6\delta < 0.32 - \frac{s_2}{2} + 6\delta + 8\varepsilon.$$

Since $\delta \leq 0.001$, we have $b_3 + c_3 \leqslant 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$. Hence Subcase **2.2** completes the proof. □

Subcase 2.5: Assume $0.066 \leq s_2 \leq 0.204$.

Lemma 4.29. *$\nu \leq 0.66$ in the case $s.2 < 0.3$ and $0.066 \leq s_2 \leq 0.204$.*

Proof. It follows from (4.3.4) and (4.3.23) that

$$a_3 \geq 0.32 - 4\delta_s - s_1 - 2\delta \geq 0.28 - 4\varepsilon - s_1 - 2\delta.$$

Thus $a_3 > 0.34 - s_1 - s_2 + \delta$, since $s_2 \geq 0.066 \geqslant 0.062 + 4\varepsilon + 3\delta$ and $\delta \leq 0.001$. It now follows from (4.3.13) that we may assume $\tau_3 = a_3 \geq 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$. Thus (4.3.6) gives $b_3, c_3 \leqslant 0.34 - a_3 + \delta < 0.01 + \frac{s_2}{2} + \frac{3\delta}{2}$, which in turn gives $b_3 + c_3 < 0.02 + s_2 + 3\delta$. Since $s_2 \leq 0.204$ and $\delta \leq 0.001$, we deduce that $b_3 + c_3 < 0.33 - \frac{s_2}{2} - \frac{\delta}{2}$. Hence Subcase **2.2** completes the proof. ☐

Subcase 2.6: Assume $2s_1 - s_2 > 0.025$.

Lemma 4.30. *$\nu \leq 0.66$ in the case $s_2 < 0.3$ and $2s_1 - s_2 > 0.025$.*

Proof.

In this case we note that the intervals in (4.3.13) overlap, since $\delta \leq 0.001$. Hence for any τ_3 belonging to the set (4.3.11), we have

$$\tau_3 \in \left(0.34 - s_1 - s_2 + \delta,\ 0.33 - \frac{\delta}{2}\right) \implies \nu < 0.66. \tag{4.3.24}$$

Furthermore, in the light of Subcases $\mathbf{S_3}$ and $\mathbf{S_4}$, we may assume that $4s_1 + 3s_2 \leq 0.71$ and $4s_1 + s_2 \geq 0.4$. In particular, these imply that $s_1 \leq \frac{0.71}{4} \leq 0.1775$ and $s_2 \leq \frac{1}{3}(0.71 - 4s_1) \leq \frac{1}{3}(0.71 - 0.4 + s_2)$, so that $s_2 \leq 0.155$. Then, on appealing to Subcase $\mathbf{S}_5$, we may assume that $s_2 < 0.066$. Similarly, it follows from Subcase $\mathbf{S}_1$ that we may also assume $a_3 < 0.32$. Thus (4.3.24) and the bound $\delta \leq 0.001$ imply that we may assume $a_3 < 0.34 - s_1 - s_2 + \delta$.

If we also had $b_3 + c_3 < 0.34 - s_1 - s_2 + \delta$, then we would have $s_3 = a_3 + b_3 + c_3 < 0.68 - 2s_1 - 2s_2 + 2\delta$. Combining this with (4.3.15), we would then conclude that

$$0.68 - 2s_1 - 2s_2 + 2\delta > s_3 \geqslant 1 - 4\delta_s - 3s_1 - 2s_2,$$

which implies that $s_1 > 0.32 - 4\delta_s - 2\delta$. Recalling (4.3.4) and the inequalities $s_1 \leq 0.1775$ and $\delta \leq 0.001$, this is a contradiction. Hence we may assume that $b_3 + c_3 \geqslant 0.34 - s_1 - s_2 + \delta$, and by (4.3.24), we may assume $\tau_3 = b_3 + c_3 \geq 0.33 - \frac{\delta}{2}$, so $b_3 > 0.165 - \frac{\delta}{4} > 0.164$. Thus we have $a_3, b_3 \in [0.164, 0.341 - s_1 - s_2]$, since $\delta \leq 0.001$. In particular the interval is nontrivial, so $s_1 + s_2 \leqslant 0.1775$.

Letting $M_i = \max(a_i, b_i)$ and $m_i = \min(a_i, b_i)$, it follows from the Fourier bound that

$$\nu < \frac{1}{2}\Big(1 + \delta + \sum_i \max(a_i, b_i) - \max(a_3, b_3)\Big) = \frac{1}{2}\Big(1 + \delta + \sum_{i \neq 3} M_i\Big).$$

It follows from (4.3.1) that $\sum_i (M_i + m_i) = \frac{2}{3} - \delta_{ab}$, and so

$$\begin{aligned} 2\nu - 1 - \delta \leqslant \sum_{i\neq 3} M_i \leqslant M_1 + M_2 + \sum_{i \geqslant 4}(M_i + m_i) \\ = M_1 + M_2 + \Big(\frac{2}{3} - \delta_{ab} - \sum_{i \leqslant 3}(M_i + m_i)\Big) \\ \leqslant \frac{2}{3} - \delta_{ab} - m_1 - m_2 - m_3 - M_3. \end{aligned}$$

By (4.3.14) we have $3a_1 + 2a_2 + a_3 \geqslant \frac{1}{3} - 4\delta_a$, and similarly, $3b_1 + 2b_2 + b_3 \geqslant \frac{1}{3} - 4\delta_b$. Thus $m_1 \geqslant \frac{1}{3}(\frac{1}{3} - 2M_2 - M_3 - 4\max(\delta_a, \delta_b))$. This together with the bounds $\delta \leq 0.001$ and (4.3.3) lead to the upper bound

$$\begin{aligned} 2\nu - 1 - \delta &\leqslant \frac{2}{3} - \delta_{ab} + \frac{1}{3}(2M_2 + M_3 - \frac{1}{3} + 4\max(\delta_a, \delta_b)) - m_2 - m_3 - M_3 \\ &\leqslant \frac{5}{9} + \frac{2}{3}M_2 + \frac{1}{3}\max(\delta_a, \delta_b) - \min(\delta_a, \delta_b) - m_3 - \frac{2}{3}M_3 \\ &\leqslant \frac{5}{9} + \frac{2}{3}M_2 + \frac{1}{3}(0.01\bar{3} + \delta + \varepsilon) + (0.00\bar{6} + \delta) - m_3 - \frac{2}{3}M_3 \\ &\leqslant 0.568 + \frac{2}{3}(M_2 - M_3) - m_3, \end{aligned}$$

using $\delta_a, \delta_b \in [-0.00\bar{6} - \delta, 0.01\bar{3} + \delta + \varepsilon]$ and $b_3 + c_3 \geqslant 0.33 - \frac{\delta}{2}$. Thus we have

$$\begin{aligned}\nu &\leqslant 0.784 + \frac{1}{3}\big(\max(a_2, b_2) - \max(a_3, b_3)\big) - \frac{1}{2}\min(a_3, b_3)\\ &\leqslant 0.784 + \frac{1}{3}\big(\max(a_2, b_2) - a_3\big) - \frac{1}{2}b_3 \qquad (4.3.25)\end{aligned}$$

Next, if $a_2 < b_2$, let $e_i := a_i$ for all $i \geqslant 1$, otherwise let $e_i := b_i$ for all $i \geqslant 1$. In particular, note $e_2 = \min(a_2, b_2)$ and $e_3 \geqslant \min(a_3, b_3) \geqslant c_3$. By a similar argument with $(a_i, b_i)_i$ replaced by $(e_i, c_i)_i$, we have

$$\begin{aligned}\nu &\leqslant 0.784 + \frac{1}{3}\big(\max(e_2, c_2) - \max(e_3, c_3)\big) - \frac{1}{2}\min(e_3, c_3)\\ &\leqslant 0.784 + \frac{1}{3}\big(\max(e_2, c_2) - e_3\big) - \frac{1}{2}c_3.\end{aligned}$$

Averaging the bound with (4.3.25), we obtain

$$\begin{aligned}\nu &\leqslant 0.784 + \frac{1}{6}\big(\max(a_2, b_2) + \max(e_2, c_2) - a_3 - e_3\big) - \frac{1}{4}(b_3 + c_3)\\ &\leqslant 0.784 + \frac{1}{6}s_2 - \frac{5}{12}(b_3 + c_3)\\ &\leqslant 0.784 + \frac{1}{6}(0.066) - \frac{5}{12}(0.33 - \tfrac{\delta}{2}) \leqslant 0.658. \qquad (4.3.26)\end{aligned}$$

Here we used $\max(a_2, b_2) + \max(e_2, c_2) = \max\big(a_2 + b_2, \max(a_2, b_2) + c_2\big) \leqslant s_2$ and $a_3 + e_3 \geqslant b_3 + c_3 \geqslant 0.33 - \frac{\delta}{2}$.

This completes the proof.

□

Theorem 4.31. $\nu \leqslant 0.66$.

Proof. Immediate from Lemmas 4.21 and 4.24. □

Proof of Theorem2.3. □

Chapter 5

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